Perturbative construction of the two-dimensional O(N) non-linear sigma model with ERG

B C Lütfüoğlu‡ and H Sonoda

Physics Department, Kobe University, Kobe 657-8501, Japan

E-mail: `bcan@itu.edu.tr`, `hsonoda@kobe-u.ac.jp`

Abstract. We use the exact renormalization group (ERG) perturbatively to construct the Wilson action for the two-dimensional O(N) non-linear sigma model. The construction amounts to regularization of a non-linear symmetry with a momentum cutoff. A quadratically divergent potential is generated by the momentum cutoff, but its non-invariance is compensated by the jacobian of the non-linear symmetry transformation.

PACS numbers: 11.10.Gh, 11.30.Na, 11.10.Kk

‡ Present address: Department of Physics, Istanbul Technical University, Turkey

1. Introduction

The two dimensional O(N) non-linear σ model is important for its asymptotic freedom and dynamical generation of a mass gap. Classically the model is defined by the action

$$S_{cl} = -\frac{1}{2g} \int d^2x \sum_{I=1}^{N} \partial_\mu \Phi_I \partial_\mu \Phi_I \tag{1}$$

where the real scalar fields satisfy the non-linear constraint $\sum_{I=1}^{N} \Phi_I \Phi_I = 1$. Regarding the model as a classical spin system, g plays the role of the temperature; large g encourages fluctuations of the fields, while small g discourages them. The asymptotic freedom of the model, first shown in [1], implies not only the validity of perturbation theory at short distances but also the generation of a mass gap due to large field fluctuations at long distances.

The purpose of this paper is to apply the method of the exact renormalization group (ERG) to renormalize the model consistently with a momentum cutoff. The perturbative renormalization of the model is done usually with the dimensional regularization [2, 3]. Its advantage is the manifest O(N) invariance, but an external magnetic field (mass term) must be introduced to avoid IR divergences. Compared with the dimensional regularization, the regularization with a momentum cutoff is physically more appealing, but it is technically more complicated; the O(N) invariance is not manifest, and a naïve sharp momentum cutoff, inconsistent with shifts of loop momenta, cannot be used beyond 1-loop.

We can overcome the technical difficulties using the formulation of field theory via ERG differential equations [4, 5]. For a general perturbative construction of theories with continuous symmetry, we refer the reader to a recent review article [6], and in this paper we give only the minimum background necessary for our purposes. ERG was first applied to the two dimensional O(N) non-linear σ model by Becchi [7]; we aim to simplify and complete his analysis. In particular, we give a perturbative algorithm for constructing the Wilson action of the model with a finite momentum cutoff Λ. The Wilson action results from an integration of fields with momenta larger than Λ, and it is free from IR divergences without an external magnetic field.

Throughout the paper we use the Euclid metric and the following notation for momentum integrals:

$$\int_p \equiv \int \frac{d^2p}{(2\pi)^2} \tag{2}$$

A short summary of this paper has appeared in sect. 6.4 of [6].

2. Momentum cutoff

We regularize the model using a UV momentum cutoff Λ_0. The bare action is given by

$$S_B = -\frac{1}{2} \int_p \frac{p^2}{K(p/\Lambda_0)} \phi_i(-p)\phi_i(p) + S_{I,B} \tag{3}$$

where the subscript i, running from 1 to $N-1$, is summed over. The interaction part is given by

$$
\begin{aligned}
S_{I,B} = \int d^2x \Big[&\Lambda_0^2 \, z_0 \left(\phi^2/2 \right) \\
&+ z_1 \left(\phi^2/2 \right) (-\partial^2) \frac{1}{2} \phi^2 + z_2 \left(\phi^2/2 \right) \phi_i(-\partial^2)\phi_i \Big]
\end{aligned} \tag{4}
$$

where we denote $\phi^2 = \phi_i \phi_i$. z_0, z_1, z_2 are functions of $\phi^2/2$ and depend logarithmically on the cutoff Λ_0. $S_{I,B}$ is the most general interaction action allowed by the manifest $O(N-1)$ invariance and perturbative renormalizability in the absence of any dimensionful parameters.

The propagator given by the free part of (3) is proportional to the smooth cutoff function $K(p/\Lambda_0)$. By choosing $K(x)$ such that

(i) $K(x)$ is a positive and non-increasing function of x^2,

(ii) $K(x) = 1$ for $x^2 < 1$,

(iii) $K(x)$ damps rapidly (faster than $1/x^2$) as $x^2 \to \infty$,

we can regularize the UV divergences of the model.

The renormalization functions $z_{0,1,2}$ must be fine tuned, first for renormalizability, and then for the $O(N)$ invariance.

3. Wilson action

The Wilson action with a finite momentum cutoff Λ has two parts:

$$
S_\Lambda \equiv S_{F,\Lambda} + S_{I,\Lambda} \tag{5}
$$

The free part

$$
S_{F,\Lambda} \equiv -\frac{1}{2} \int_p \frac{p^2}{K(p/\Lambda)} \phi_i(-p)\phi_i(p) \tag{6}
$$

gives the propagator with a finite momentum cutoff Λ:

$$
\langle \phi_i(p)\phi_j(-p) \rangle_{S_{F,\Lambda}} = \delta_{ij} \frac{K(p/\Lambda)}{p^2} \tag{7}
$$

The interaction part of the Wilson action is defined by

$$
\begin{aligned}
\exp \left[S_{I,\Lambda}[\phi] \right] &\equiv \int [d\phi'] \\
&\times \exp \left[-\frac{1}{2} \int_p \frac{p^2}{K(p/\Lambda_0) - K(p/\Lambda)} \phi_i'(-p)\phi_i'(p) + S_{I,B}[\phi + \phi'] \right] \\
&= \exp \left[\frac{1}{2} \int_p \frac{K(p/\Lambda_0) - K(p/\Lambda)}{p^2} \frac{\delta^2}{\delta\phi_i(p)\delta\phi_i(-p)} \right] \cdot \exp \left[S_{I,B}[\phi] \right]
\end{aligned} \tag{8}
$$

Alternatively, we can define $S_{I,\Lambda}$ by the differential equation [4, 5]

$$
-\Lambda \frac{\partial}{\partial \Lambda} S_{I,\Lambda} = \frac{1}{2} \int_p \frac{\Delta(p/\Lambda)}{p^2} \left\{ \frac{\delta S_{I,\Lambda}}{\delta\phi_i(-p)} \frac{\delta S_{I,\Lambda}}{\delta\phi_i(p)} + \frac{\delta^2 S_{I,\Lambda}}{\delta\phi_i(-p)\delta\phi_i(p)} \right\} \tag{9}
$$

and the initial condition

$$S_{I,\Lambda}\Big|_{\Lambda=\Lambda_0} = S_{I,B} \tag{10}$$

For a fixed Λ, we expand $S_{I,\Lambda}$ up to two derivatives to obtain

$$S_{I,\Lambda} = \int d^2x \left[\Lambda^2 a \left(\ln \Lambda/\mu; \phi^2/2 \right) + A \left(\ln \Lambda/\mu; \phi^2/2 \right) (-\partial^2) \frac{1}{2} \phi^2 \right.$$
$$\left. + B \left(\ln \Lambda/\mu; \phi^2/2 \right) \phi_i(-\partial^2)\phi_i \right] + \cdots \tag{11}$$

where the dotted part contains four or more derivatives. a, A, B are functions of $\phi^2/2$, and they can be expanded as

$$\begin{cases} a \left(\ln \Lambda/\mu; \phi^2/2 \right) &= \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\phi^2}{2} \right)^n a_n(\ln \Lambda/\mu) \\ A \left(\ln \Lambda/\mu; \phi^2/2 \right) &= \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\phi^2}{2} \right)^n A_n(\ln \Lambda/\mu) \\ B \left(\ln \Lambda/\mu; \phi^2/2 \right) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\phi^2}{2} \right)^n B_n(\ln \Lambda/\mu) \end{cases} \tag{12}$$

The Taylor coefficients depend logarithmically on the cutoff Λ. We have chosen the ratio of Λ to an arbitrary renormalization scale μ as the argument of the logarithm. The initial condition (10) gives

$$\begin{cases} a(\ln \Lambda_0/\mu; \phi^2/2) &= z_0(\phi^2/2) \\ A(\ln \Lambda_0/\mu; \phi^2/2) &= z_1(\phi^2/2) \\ B(\ln \Lambda_0/\mu; \phi^2/2) &= z_2(\phi^2/2) \end{cases} \tag{13}$$

The renormalization functions $z_{0,1,2}$ are determined so that

$$\lim_{\Lambda_0 \to \infty} S_{I,\Lambda} \tag{14}$$

exists for any finite Λ. Using the BPHZ renormalization scheme adapted to the Wilson action [8, 9, 10], we can choose $A(0; \phi^2/2)$ & $B(0; \phi^2/2)$ as any functions. As will be explained in the next section, the O(N) invariance constrains the choice of $A(0; \phi^2/2)$ & $B(0; \phi^2/2)$.

Alternatively, we can construct the continuum limit (14) directly without starting from a bare action. We demand that the dotted part of (11) is multiplied by the inverse powers of Λ. For given $A(0; \phi^2/2)$ & $B(0; \phi^2/2)$, the differential equation (9) uniquely determines $a \left(\ln \Lambda/\mu; \phi^2/2 \right)$ and the dotted part of (11) [8, 9, 10]. This is the preferred approach we adopt in the rest of the paper. In Appendix A, we summarize the basic properties of the correlation functions calculated with S_Λ.

4. WT identity for O(N)

The Wilson action is determined uniquely in terms of $A(0; \phi^2/2)$ & $B(0; \phi^2/2)$. For the O(N) symmetry, we must choose $A(0; \phi^2/2)$ & $B(0; \phi^2/2)$ appropriately. In this and the following two sections, we aim to complete the analysis of Becchi given in sect. 6 of [7].

The Wilson action has manifest O(N−1) invariance. To insure the full O(N) invariance, we must demand the invariance of the action under the following infinitesimal transformation:

$$\delta\phi_i(p) = K \left(p/\Lambda \right) \epsilon_i [\Phi_N](p) \tag{15}$$

where ϵ_i is an infinitesimal constant, and $[\Phi_N]$ is the composite operator for the N-th component of the O(N) vector, whose i-th component is proportional to ϕ_i. More precisely, $[\Phi_N]$ is defined by the ERG differential equation

$$- \Lambda \frac{\partial}{\partial \Lambda} [\Phi_N](p)$$
$$= \int_q \frac{\Delta(q/\Lambda)}{q^2} \left\{ \frac{\delta S_{I,\Lambda}}{\delta \phi_i(q)} \frac{\delta}{\delta \phi_i(q)} + \frac{1}{2} \frac{\delta^2}{\delta \phi_i(q) \delta \phi_i(-q)} \right\} [\Phi_N](p) \tag{16}$$

and the derivative expansion

$$\int_p e^{ipx} [\Phi_N](p) = P\left(\ln \Lambda/\mu; \phi(x)^2/2 \right) + \cdots \tag{17}$$

where the dotted part, proportional to the inverse powers of Λ, contains derivatives of $\phi_i(x)$. $[\Phi_N]$ is parameterized by a function

$$P\left(0; \phi^2/2 \right) \tag{18}$$

which is arbitrary as far as perturbative renormalizability of $[\Phi_N]$ is concerned. (The composite operators, the concept of which was first introduced in sect. 5 of [7], can be considered as infinitesimal deformations of the Wilson action, and they satisfy the same linear ERG differential equation as (16). A composite operator vanishes identically if the leading part in the derivative expansion, the part multiplied by the non-negative powers of Λ, vanishes. For more details, see sect. 4 of [6].)

Following Becchi [7], we now define the Ward-Takahashi (WT) composite operator for (15) by

$$\Sigma_\Lambda \equiv \int_p \left[\frac{\delta S_\Lambda}{\delta \phi_i(p)} \delta \phi_i(p) + \frac{\delta}{\delta \phi_i(p)} \delta \phi_i(p) \right]$$
$$= \epsilon_i \int_p K\left(p/\Lambda \right) \left[\frac{\delta S_\Lambda}{\delta \phi_i(p)} [\Phi_N](p) + \frac{\delta[\Phi_N](p)}{\delta \phi_i(p)} \right] \tag{19}$$

This satisfies the same ERG linear differential equation as (16). The WT identity

$$\Sigma_\Lambda = 0 \tag{20}$$

is the "quantum" invariance of the Wilson action under (15), whereby the non-trivial jacobian of (15) is taken into account. Concrete loop calculations show that the coefficient function $a(\ln \Lambda/\mu; \phi^2/2)$, corresponding to the quadratically divergent potential in the bare action, is non-vanishing. But its non-invariance under (15) is cancelled by the jacobian.

Taking the correlation of Σ_Λ with the elementary fields, we obtain the usual WT identity from (20):

$$\sum_{j=1}^n \epsilon_{i_j} \left\langle \phi_{i_1}(p_1) \cdots \Phi_N(p_j) \cdots \phi_{i_n}(p_n) \right\rangle^\infty = 0 \tag{21}$$

where the renormalized correlation function

$$\left\langle \phi_{i_1}(p_1) \cdots \Phi_N(p_j) \cdots \phi_{i_n}(p_n) \right\rangle^\infty$$
$$\equiv \prod_{k \neq j} \frac{1}{K\left(p_k/\Lambda \right)} \times \left\langle \phi_{i_1}(p_1) \cdots [\Phi_N](p_j) \cdots \phi_{i_n}(p_n) \right\rangle_{S_\Lambda} \tag{22}$$

is independent of the cutoff Λ. (This Λ independence is a consequence of the differential equations (9, 16). See, for example, sect. 4.1 of [6] for more explanations.)

For S_Λ to satisfy (20), we must fine tune not only $A(0; \phi^2/2)$ & $B(0; \phi^2/2)$ but also $P(0; \phi^2/2)$. In the next two sections, we will show the possibility of such fine tuning.

5. Tree level

We expand $S_\Lambda, S_{I,\Lambda}$, etc., in the number of loops. We use a superscript $^{(l)}$ to denote the l-loop level:

$$
\begin{aligned}
&S_\Lambda = \sum_{l=0}^\infty S_\Lambda^{(l)}, \quad [\Phi_N](p) = \sum_{l=0}^\infty [\Phi_N]^{(l)}(p), \quad \Sigma_\Lambda = \sum_{l=0}^\infty \Sigma_\Lambda^{(l)}, \\
&A\left(\ln \Lambda/\mu; \phi^2/2\right) = \sum_{l=0}^\infty A^{(l)}\left(\ln \Lambda/\mu; \phi^2/2\right), \cdots \\
&A_i\left(\ln \Lambda/\mu\right) = \sum_{l=0}^\infty A_i^{(l)}\left(\ln \Lambda/\mu\right), \cdots \\
&A\left(0; \phi^2/2\right) = \sum_{l=0}^\infty A^{(l)}\left(\phi^2/2\right), \cdots
\end{aligned}
\tag{23}
$$

In this section, we show how to tune

$$
A^{(0)}\left(\phi^2/2\right), \quad B^{(0)}\left(\phi^2/2\right), \quad P^{(0)}\left(\phi^2/2\right)
\tag{24}
$$

to satisfy $\Sigma_\Lambda^{(0)} = 0$.

The leading part of the derivative expansion of $S_\Lambda^{(0)}$ is given by the classical action:

$$
S_\Lambda^{(0)} = S_{cl} + \cdots
\tag{25}
$$

S_{cl} is independent of Λ, and we can write

$$
\begin{aligned}
S_{cl} = \int d^2x \Big[&-\frac{1}{2}\partial_\mu \phi_i \partial_\mu \phi_i \\
&+ A^{(0)}\left(\phi^2/2\right)(-\partial^2)\frac{\phi^2}{2} + B^{(0)}\left(\phi^2/2\right)\phi_i(-\partial^2)\phi_i \Big]
\end{aligned}
\tag{26}
$$

Likewise, the derivative expansion of $[\Phi_N]^{(0)}$ gives

$$
\int_p e^{ipx}[\Phi_N]^{(0)}(p) = P^{(0)}\left(\phi^2/2\right) + \cdots
\tag{27}
$$

As a convention, we can choose

$$
A_0^{(0)} = B_0^{(0)} = 0, \quad P_0^{(0)} = 1
\tag{28}
$$

At tree level, the WT identity gives

$$
\Sigma_\Lambda^{(0)} \equiv \epsilon_i \int_p K\left(p/\Lambda\right) \frac{\delta S_\Lambda^{(0)}}{\delta \phi_i(p)}[\Phi_N]^{(0)}(p) = 0
\tag{29}
$$

The derivative expansion gives

$$
\Sigma_{cl} \equiv \epsilon_i \int d^2x \, \frac{\delta S_{cl}}{\delta \phi_i(x)} P^{(0)}\left(\phi(x)^2/2\right) = 0
\tag{30}
$$

Substituting (26) into the above, we obtain

$$
\begin{aligned}
\Sigma_{cl} = \epsilon_i \int d^2x \, \phi_i \Big[&\partial_\mu \phi_j \partial_\mu \phi_j \left\{ P^{(0)\prime} - (2A^{(0)\prime} + B^{(0)\prime})P^{(0)} - 2P^{(0)\prime}B^{(0)} \right\} \\
&+ \phi_j \partial^2 \phi_j \left\{ P^{(0)\prime} - 2(A^{(0)\prime} + B^{(0)\prime})P^{(0)} - 2P^{(0)\prime}B^{(0)} \right\} \\
&+ (\phi_j \partial_\mu \phi_j)^2 \left\{ (1 - 2B^{(0)})P^{(0)\prime\prime} - (A^{(0)\prime\prime} + B^{(0)\prime\prime})P^{(0)} - 2B^{(0)\prime}P^{(0)\prime} \right\} \Big]
\end{aligned}
\tag{31}
$$

where the prime denotes a derivative with respect to $\phi^2/2$. For Σ_{cl} to vanish, we must satisfy the following three equations:

$$(1 - 2B^{(0)})P^{(0)'} - (2A^{(0)'} + B^{(0)'})P^{(0)} = 0 \qquad (32a)$$

$$(1 - 2B^{(0)})P^{(0)'} - 2(A^{(0)'} + B^{(0)'})P^{(0)} = 0 \qquad (32b)$$

$$(1 - 2B^{(0)})P^{(0)''} - 2B^{(0)'}P^{(0)'} - (A^{(0)''} + B^{(0)''})P^{(0)} = 0 \qquad (32c)$$

From $(32a)$ and $(32b)$, we get

$$B^{(0)'}(x)P^{(0)}(x) = 0 \qquad (33)$$

where we write $x \equiv \phi^2/2$ for short. Since $P^{(0)}(x) \neq 0$, we obtain $B^{(0)'}(x) = 0$; hence using (28) we obtain

$$B^{(0)}(x) = 0 \qquad (34)$$

Thus, $(32a)$ gives

$$P^{(0)'}(x) = 2A^{(0)'}(x)P^{(0)}(x) \qquad (35)$$

Using (28), we obtain

$$P^{(0)}(x) = \exp\left[2A^{(0)}(x)\right] \qquad (36)$$

Finally, $(32c)$ gives

$$P^{(0)''}(x) = A^{(0)''}(x)P^{(0)}(x) \qquad (37)$$

This is solved by

$$A^{(0)}(x) = \frac{1}{4}\ln(1 - 2c\,x) \qquad (38)$$

where c is an arbitrary constant. Hence, we obtain

$$P^{(0)}(x) = \sqrt{1 - 2cx} \qquad (39)$$

The constant c may be chosen either positive or negative. If we choose a positive $c = g > 0$, then we obtain

$$\Phi_N = \sqrt{1 - g\phi^2} \qquad (40)$$

appropriate for the classical O(N) non-linear σ model. If we choose a negative $c = -g < 0$ instead, we obtain

$$\Phi_N = \sqrt{1 + g\phi^2} \qquad (41)$$

appropriate for the classical O(N−1,1) non-linear σ model. We make the first choice.

To summarize, we have obtained

$$\begin{cases} A^{(0)}(x) &= \frac{1}{4}\ln(1 - 2gx) \\ B^{(0)}(x) &= 0 \\ P^{(0)}(x) &= \sqrt{1 - 2gx} \end{cases} \qquad (42)$$

where g is an arbitrary **positive** coupling constant. The corresponding classical action is given by the familiar expression

$$S_{cl} = -\frac{1}{2g}\int d^2x\left[g\partial_\mu\phi_i\partial_\mu\phi_i + \partial_\mu\sqrt{1 - g\phi^2}\cdot\partial_\mu\sqrt{1 - g\phi^2}\right] \qquad (43)$$

6. Loop levels

Let us now assume that we have determined S_Λ and $[\Phi_N]$ up to l-loop level ($l \geq 0$) such that

$$\Sigma_\Lambda^{(0)} = \cdots = \Sigma_\Lambda^{(l)} = 0 \tag{44}$$

Under this induction hypothesis, we wish to determine $S_\Lambda^{(l+1)}$ and $[\Phi_N]^{(l+1)}$ (or equivalently $A^{(l+1)}(x), B^{(l+1)}(x), P^{(l+1)}(x)$) such that

$$\Sigma_\Lambda^{(l+1)} = 0 \tag{45}$$

Note that Σ_Λ is a composite operator, satisfying the same ERG differential equation as (16). Applying the loop expansion and using the induction hypothesis, we find

$$-\Lambda \frac{\partial}{\partial \Lambda} \Sigma_\Lambda^{(l+1)} = \int_p \frac{\Delta(p/\Lambda)}{p^2} \frac{\delta S_{I,\Lambda}^{(0)}}{\delta \phi_i(-p)} \frac{\delta \Sigma_\Lambda^{(l+1)}}{\delta \phi_i(p)} \tag{46}$$

Calling the leading part of the derivative expansion of $\Sigma_\Lambda^{(l+1)}$ by $\tilde{\Sigma}^{(l+1)}$, we obtain

$$-\Lambda \frac{\partial}{\partial \Lambda} \tilde{\Sigma}^{(l+1)} = 0 \tag{47}$$

since $\Delta(p/\Lambda) = 0$ for $p^2 < \Lambda^2$. Hence, $\tilde{\Sigma}^{(l+1)}$ is independent of Λ. Thus, we obtain

$$\tilde{\Sigma}^{(l+1)}[\phi] = \epsilon_i \int d^2x \, \phi_i \left[\partial_\mu \phi_j \partial_\mu \phi_j \cdot s_1 \left(\phi^2/2 \right) \right.$$
$$\left. + \phi_j \partial^2 \phi_j \cdot s_2 \left(\phi^2/2 \right) + (\phi_j \partial_\mu \phi_j)^2 \cdot s_3 \left(\phi^2/2 \right) \right] \tag{48}$$

where $s_i(\phi^2/2)$ ($i = 1, 2, 3$) are functions of $\phi^2/2$, independent of $\ln \Lambda/\mu$.

The definition (20) of Σ_Λ gives the decomposition

$$\Sigma_\Lambda^{(l+1)} = \Sigma_\Lambda^{(l+1),t} + \Sigma_\Lambda^{(l+1),u} \tag{49}$$

where

$$\Sigma_\Lambda^{(l+1),t} = \epsilon_i \int_p K\left(p/\Lambda\right) \left[\frac{\delta S_\Lambda^{(l+1)}}{\delta \phi_i(p)} [\Phi_N]^{(0)}(p) + \frac{\delta S_\Lambda^{(0)}}{\delta \phi_i(p)} [\Phi_N]^{(l+1)}(p) \right] \tag{50a}$$

$$\Sigma_\Lambda^{(l+1),u} = \epsilon_i \int_p K\left(p/\Lambda\right) \left[\sum_{k=1}^{l} \frac{\delta S_\Lambda^{(k)}}{\delta \phi_i(p)} [\Phi_N]^{(l+1-k)}(p) + \frac{\delta[\Phi_N]^{(l)}(p)}{\delta \phi_i(p)} \right] \tag{50b}$$

Only $\Sigma_\Lambda^{(l+1),t}$ depends on $A^{(l+1)}(x), B^{(l+1)}(x), P^{(l+1)}(x)$, and $\Sigma_\Lambda^{(l+1),u}$ are determined by S_Λ and $[\Phi_N]$ up to l-loop.

Therefore, the functions $s_i(x)$ are given as the sum

$$s_i(x) = t_i(x) + u_i(x) \quad (i = 1, 2, 3) \tag{51}$$

where $t_i(x)$ are linear in $A^{(l+1)}(x), B^{(l+1)}(x), P^{(l+1)}(x)$, and $u_i(x)$ are determined by the lower loop functions. We obtain explicitly

$$t_1(x) = P^{(l+1)'} - (2A^{(l+1)'} + B^{(l+1)'})P^{(0)}$$
$$- 2A^{(0)'} P^{(l+1)} - 2P^{(0)'} B^{(l+1)} \tag{52a}$$
$$t_2(x) = P^{(l+1)'} - 2(A^{(l+1)'} + B^{(l+1)'})P^{(0)}$$

$$- 2A^{(0)'} P^{(l+1)} - 2P^{(0)'} B^{(l+1)} \tag{52b}$$

$$t_3(x) = P^{(l+1)''} - (A^{(l+1)''} + B^{(l+1)''})P^{(0)}$$
$$- A^{(0)''} P^{(l+1)} - 2P^{(0)''} B^{(l+1)} - 2B^{(l+1)'} P^{(0)'} \tag{52c}$$

There is no relation among the $t(x)$'s. Thus, whatever $u(x)$'s are, we can solve the equations

$$s_i(x) = t_i(x) + u_i(x) = 0 \quad (i = 1, 2, 3) \tag{53}$$

Using (42), the solution is obtained explicitly as follows:

$$B^{(l+1)}(x) = B^{(l+1)}(0) + \int_0^x dy \frac{-u_1(y) + u_2(y)}{\sqrt{1 - 2gy}} \tag{54a}$$

$$\frac{d}{dx} A^{(l+1)}(x) = \frac{1}{(1 - 2gx)^2} \left[A^{(l+1)'}(0) \right.$$
$$+ \int_0^x dy \left\{ -2g^2 B^{(l+1)}(y) + (1 - 2gy) B^{(l+1)''}(y) \right.$$
$$+ g\sqrt{1 - 2gy} \left(-2u_1(y) + u_2(y) \right)$$
$$\left. + (1 - 2gy)^{\frac{3}{2}} \left(2u_1'(y) - u_2'(y) - u_3(y) \right) \right\} \left. \right] \tag{54b}$$

$$P^{(l+1)}(x) = \sqrt{1 - 2gx} \left[P^{(l+1)}(0) + \int_0^x dy \left\{ 2A^{(l+1)'}(y) \right. \right.$$
$$\left. \left. - \frac{2g}{1 - 2gy} B^{(l+1)}(y) + \frac{-2u_1(y) + u_2(y)}{\sqrt{1 - 2gy}} \right\} \right] \tag{54c}$$

Note that

$$A^{(l+1)'}(0), \quad B^{(l+1)}(0), \quad P^{(l+1)}(0) \tag{55}$$

are left undetermined as constants of integration. This is expected, since $A^{(l+1)'}(0)$ normalizes the coupling g, $B^{(l+1)}(0)$ normalizes the field ϕ^i, and $P^{(l+1)}(0)$ normalizes the composite operator $[\Phi_N]$. For example, we can adopt the convention [8, 9, 10]

$$A_1 \left(\ln \Lambda/\mu \right) \Big|_{\Lambda=\mu} = \frac{\partial}{\partial x} A(0; x) \Big|_{x=0} = -\frac{g}{2} \tag{56a}$$

$$B_0 \left(\ln \Lambda/\mu \right) \Big|_{\Lambda=\mu} = B(0; 0) = 0 \tag{56b}$$

$$P_0 \left(\ln \Lambda/\mu \right) \Big|_{\Lambda=\mu} = P(0; 0) = 1 \tag{56c}$$

analogous to the minimal subtraction for dimensional regularization [11]. This concludes our inductive construction of the O(N) non-linear σ model.

7. 1-loop results

Let us give explicitly the 1-loop corrections to 2- and 4-point vertices in the Wilson action. (A little more details are given in Appendix B.) For the 2-point vertex, we find

$$a_1^{(1)} = \frac{g}{2} \int_q \Delta(q) = g \int_q K(q) \tag{57a}$$

$$B_0^{(1)} = \frac{g}{4\pi} \ln \Lambda/\mu \tag{57b}$$

and for the 4-point vertex, we find

$$a_2^{(1)} = 2g^2 \int_q \Delta(q) K(q) = 2g^2 \int_q K(q)^2 \tag{58a}$$

$$A_1^{(1)} = N \frac{g^2}{4\pi} \ln \Lambda/\mu \tag{58b}$$

$$B_1^{(1)} = \text{const} \tag{58c}$$

We also find the 2- and 4-point vertices for the composite operator $[\Phi_N](p)$:

$$P_0^{(1)} = \frac{N-1}{4\pi} g \ln \Lambda/\mu \tag{59a}$$

$$P_1^{(1)} = (N-1) \frac{g^2}{4\pi} \ln \Lambda/\mu + \text{const} \tag{59b}$$

We have fixed the Λ independent part of $A_1^{(1)}$, $B_0^{(1)}$, and $P_0^{(1)}$ using the convention ($56a$, $56b$, $56c$).

The two constants in $B_1^{(1)}$ and $P_1^{(1)}$ are left undetermined by the ERG differential equations. They are determined by the WT identity. Calculating

$$\epsilon_i \int_p K\left(p/\Lambda\right) \frac{\delta[\Phi_N]^{(0)}(p)}{\delta\phi_i(p)} \tag{60}$$

only up to cubic in fields and up to two derivatives, we obtain

$$u_1^{(1)}(0) = -g^2 \left(\int_q \frac{K(q)(1-K(q))}{q^2} + \frac{1}{4\pi} \right) \tag{61a}$$

$$u_2^{(1)}(0) = g^2 \int_q \frac{K(q)}{q^2} \left(\frac{1}{4} \tilde{\Delta}(q) - 2(1-K(q)) \right) \tag{61b}$$

where

$$\tilde{\Delta}(q) \equiv -2q^2 \frac{d}{dq^2} \Delta(q) \tag{62}$$

Hence, we obtain

$$B_1^{(1)} = g^2 \left(\frac{1}{4\pi} - \int_q \frac{K(1-K)}{q^2} + \frac{1}{4} \int_q \frac{K\tilde{\Delta}}{q^2} \right) \tag{63a}$$

$$P_1^{(1)} \Big|_{\Lambda=\mu} = g^2 \left(\frac{1}{2\pi} + \frac{1}{4} \int_q \frac{K\tilde{\Delta}}{q^2} \right) \tag{63b}$$

In Appendix C we explain how to obtain the beta function of g and anomalous dimension of ϕ_i in the ERG approach. The above 1-loop results reproduce the well known results first obtained in [1]:

$$\beta(g) \simeq (N-2) \frac{g^2}{2\pi}, \quad \gamma(g) \simeq \frac{g}{4\pi} \tag{64}$$

8. Concluding remarks

In this paper we have applied the ERG formulation of quantum field theory for the perturbative construction of the two-dimensional non-linear σ model. The model is

parameterized by three renormalization functions $A(0; x), B(0; x), P(0; x)$, and we have shown how to tune these by imposing the WT identity (20). Only short-distance physics can be explored perturbatively, and long-distance physics needs non-trivial approximations, such as $1/N$. For the $1/N$ expansions it is common to linearize the O(N) symmetry using an auxiliary field; it would be interesting to extend the ERG formulation to accommodate the auxiliary field.

Appendix A. Basic properties of the correlation functions

The correlation functions of a Wilson action S_Λ are dependent on the cutoff Λ. Using the inverse of the cutoff function, however, we can easily construct Λ independent correlation functions:

$$
\begin{aligned}
\langle \phi_i(p)\phi_j(-p)\rangle^\infty \quad &\equiv \frac{1}{K\left(p/\Lambda\right)^2}\langle \phi_i(p)\phi_j(-p)\rangle_{S_\Lambda} \\
&\quad + \delta_{ij}\frac{1 - 1/K\left(p/\Lambda\right)}{p^2}
\end{aligned}
\tag{A.1}
$$

$$
\langle \phi_{i_1}(p_1)\cdots\phi_{i_n}(p_n)\rangle^\infty \equiv \prod_{j=1}^{n}\frac{1}{K\left(p_j/\Lambda\right)}\cdot\langle \phi_{i_1}(p_1)\cdots\phi_{i_n}(p_n)\rangle_{S_\Lambda}
\tag{A.2}
$$

for $n \geq 4$. The Λ independence of these correlation functions is a consequence of the ERG differential equation (9). See sect. 2 of [6] for more details.

Similarly, given a composite operator $\mathcal{O}_\Lambda$ that satisfies the same linear ERG differential equation as (16), we can construct Λ independent correlation functions by

$$
\langle \mathcal{O}\,\phi_{i_1}(p_1)\cdots\phi_{i_n}(p_n)\rangle^\infty \equiv \prod_{j=1}^{n}\frac{1}{K\left(p_j/\Lambda\right)}\cdot\langle \mathcal{O}_\Lambda\,\phi_{i_1}(p_1)\cdots\phi_{i_n}(p_n)\rangle_{S_\Lambda}
\tag{A.3}
$$

See sect. 4 of [6] for more details.

Appendix B. 1-loop calculations

The interaction part of the classical action is given by

$$
\begin{aligned}
S_{I,cl} &\equiv \int d^2x(-\partial^2)\frac{\phi^2}{2}\cdot\frac{1}{4}\ln\left(1 - g\phi^2\right) \\
&= \int d^2x(-\partial^2)\frac{\phi^2}{2}\cdot\frac{-1}{4}\sum_{n=1}^{\infty}(2g)^n(n-1)!\cdot\frac{1}{n!}\left(\frac{\phi^2}{2}\right)^n
\end{aligned}
\tag{B.1}
$$

Hence,

$$
A_n^{(0)} = -\frac{1}{4}(n-1)!(2g)^n
\tag{B.2}
$$

Thus, for the graph in Figure B1, we obtain the Feynman rule

$$
\delta_{i_1 i_2}\cdots\delta_{i_{2n-1}i_{2n}}\left\{(p_1 + p_2)^2 + \cdots + (p_{2n-1} + p_{2n})^2\right\}A_{n-1}^{(0)}
\tag{B.3}
$$

As the simplest example, we consider the 1-loop contribution to the two-point vertex given by the Feynman graph in Figure B2. The ERG differential equation gives

Figure B1. Tree level vertex $(n \geq 2)$

Figure B2. $\mathcal{V}_2^{(1)}(p)$: 1-loop correction to the two-point vertex with momentum p

$$-\Lambda \frac{\partial}{\partial \Lambda} \mathcal{V}_2^{(1)}(p) = \int_q \frac{\Delta(q/\Lambda)}{q^2} 2(p+q)^2 A_1^{(0)} = -\frac{g}{2} \int_q \frac{\Delta(q/\Lambda)}{q^2} 2(p+q)^2$$

$$= -g \int_q \frac{\Delta(q/\Lambda)}{q^2}(p^2+q^2) = -g \left[\Lambda^2 \int_q \Delta(q) + \frac{1}{2\pi} p^2 \right] \text{(B.4)}$$

where we have used

$$\int_q \frac{\Delta(q)}{q^2} = \frac{1}{2\pi} \tag{B.5}$$

Hence, integrating this over Λ, we obtain

$$\mathcal{V}_2^{(1)}(p) = \Lambda^2 \frac{g}{2} \int_q \Delta(q) + 2p^2 \frac{g}{4\pi} \ln \Lambda/\mu \tag{B.6}$$

This gives

$$a_1^{(1)} = \frac{g}{2} \int_q \Delta(q), \quad B_0^{(1)} (\ln \Lambda/\mu) = \frac{g}{4\pi} \ln \Lambda/\mu \tag{B.7}$$

where we have used the normalization condition $B_0^{(1)}(0) = 0$.

As another example, let us consider the 1-loop contribution to the 1-point vertex of the jacobian:

$$\int_q K(q/\Lambda) \frac{\delta[\Phi_N]^{(0)}(q)}{\delta \phi_i(q)} \tag{B.8}$$

Now, the leading part of the derivative expansion of $[\Phi_N]^{(0)}$ is given by

$$\int_p e^{ipx} [\Phi_N]^{(0)}(p) = P^{(0)} \left(\phi(x)^2/2 \right) + \cdots \tag{B.9}$$

where

$$P^{(0)}(x) = \sqrt{1-2gx} = 1 - \sum_{n=1}^{\infty} \frac{(2n-2)!}{2^{n-1}(n-1)!} g^n \cdot \frac{x^n}{n!} \tag{B.10}$$
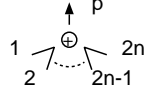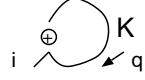
Hence, we obtain

$$P_0^{(0)} = 1, \quad P_{n\geq 1}^{(0)} = -\frac{(2n-2)!}{2^{n-1}(n-1)!} g^n \tag{B.11}$$

Let us denote the $2n$-point vertex $P_n^{(0)}$ for $[\Phi_N]$ by Figure B3. Then, the one-point vertex for the 1-loop jacobian is given by Figure B4 and calculated as

$$\int_q K(q/\Lambda) P_1^{(0)} = -\Lambda^2 g \int_q K(q) = -\Lambda^2 \frac{g}{2} \int_q \Delta(q) \tag{B.12}$$

This cancels the contribution of the $a_1^{(1)}$ term to $\Sigma_\Lambda^{(1)}$.

Figure B3. $2n$-point vertex for $[\Phi_N](p)$

Figure B4. 1-loop contribution to the 1-point vertex of the jacobian (B.8)

Appendix C. Beta function and anomalous dimension

The derivation of the mass independent beta functions and anomalous dimensions in the ERG formalism has been discussed in [8] and [10].

Appendix C.1. μ dependence of the Wilson action

The Wilson action S_Λ for a different choice of μ satisfies the same ERG differential equation (9). Hence,

$$\Psi_\Lambda \equiv -\mu \partial_\mu S_\Lambda \tag{C.1}$$

is a composite operator satisfying the ERG differential equation

$$-\Lambda \frac{\partial}{\partial \Lambda} \Psi_\Lambda = \int_p \frac{\Delta(p/\Lambda)}{p^2} \left\{ \frac{\delta S_{I,\Lambda}}{\delta \phi_i(p)} \frac{\delta \Psi_\Lambda}{\delta \phi_i(-p)} + \frac{1}{2} \frac{\delta^2 \Psi_\Lambda}{\delta \phi_i(p) \delta \phi_i(-p)} \right\} \tag{C.2}$$

Ψ_Λ has the correlation functions

$$\langle \Psi \, \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty = -\mu \partial_\mu \langle \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty \tag{C.3}$$

Expanding Ψ_Λ up to two derivatives, we obtain

$$\Psi_\Lambda = \int d^2x \left[\Lambda^2 \, \dot{a}(\ln \Lambda/\mu; \phi^2/2) + \dot{A}(\ln \Lambda/\mu; \phi^2/2) \, (-\partial^2) \frac{\phi^2}{2} \right.$$
$$\left. + \dot{B}(\ln \Lambda/\mu; \phi^2/2) \, \phi_i(-\partial^2)\phi_i \right] + \cdots \tag{C.4}$$

where

$$\begin{cases} \dot{a}(\ln \Lambda/\mu; x) &\equiv \frac{\partial}{\partial \ln \Lambda/\mu} a(\ln \Lambda/\mu; x) \\ \dot{A}(\ln \Lambda/\mu; x) &\equiv \frac{\partial}{\partial \ln \Lambda/\mu} A(\ln \Lambda/\mu; x) \\ \dot{B}(\ln \Lambda/\mu; x) &\equiv \frac{\partial}{\partial \ln \Lambda/\mu} B(\ln \Lambda/\mu; x) \end{cases} \tag{C.5}$$

Especially at $\Lambda = \mu$, the coefficient of $(\phi^2/2)(-\partial^2)(\phi^2/2)$ is

$$\partial_x \dot{A}(0; x) \Big|_{x=0} \tag{C.6}$$

and that of $\phi_i(-\partial^2)\phi_i$ is

$$\dot{B}(0; 0) \tag{C.7}$$

Since Ψ_Λ is an infinitesimal change of the Wilson action, it has two degrees of freedom, corresponding to the infinitesimal variation of g and that of the normalization of ϕ_i. Thus, we can construct two composite operators:

(i) $\mathcal{O}_g$ that generates an infinitesimal change of g:

$$\mathcal{O}_g \equiv -\partial_g S_\Lambda \tag{C.8}$$

The correlation functions of $\mathcal{O}_g$ are given by

$$\langle \mathcal{O}_g \, \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty = -\partial_g \langle \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty \tag{C.9}$$

(ii) $\mathcal{N}$ that generates an infinitesimal renormalization of ϕ_i:

$$\mathcal{N} \equiv - \int_p \phi_i(p) \frac{\delta S_\Lambda}{\delta \phi_i(p)} - \int_p \frac{K(p/\Lambda)(1 - K(p/\Lambda))}{p^2}$$
$$\times \left\{ \frac{\delta S_\Lambda}{\delta \phi_i(p)} \frac{\delta S_\Lambda}{\delta \phi_i(-p)} + \frac{\delta^2 S_\Lambda}{\delta \phi_i(p)\delta \phi_i(-p)} \right\} \tag{C.10}$$

$\mathcal{N}$ counts the number of fields:

$$\langle \mathcal{N} \, \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty = n \langle \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty \tag{C.11}$$

Ψ_Λ must be a linear combination of $\mathcal{O}_g$ and $\mathcal{N}$; hence

$$\Psi_\Lambda = \beta(g)\,\mathcal{O}_g + \gamma(g)\,\mathcal{N} \tag{C.12}$$

where neither $\beta(g)$ nor $\gamma(g)$ depends on Λ. This gives the differential equation

$$(-\mu\partial_\mu + \beta\partial_g) \langle \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty = n\gamma \langle \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty \tag{C.13}$$

Hence, $\beta(g)$ is the beta function of g, and $\gamma(g)$ is the anomalous dimension of ϕ_i.

Appendix C.2. $\mathcal{O}_g$

The derivative expansion gives

$$\mathcal{O}_g = \int d^2x \left[\Lambda^2(-\partial_g)\, a(\ln \Lambda/\mu; \phi^2/2) + (-\partial_g)A(\ln \Lambda/\mu; \phi^2/2)\,(-\partial^2)\frac{\phi^2}{2} \right.$$
$$\left. + (-\partial_g)B(\ln \Lambda/\mu; \phi^2/2)\,\phi_i(-\partial^2)\phi_i \right] + \cdots \tag{C.14}$$

At $\Lambda = \mu$, the coefficient of $(\phi^2/2)(-\partial^2)(\phi^2/2)$ is

$$-\partial_g\partial_x A(0; x)\Big|_{x=0} = \frac{1}{2} \tag{C.15}$$

and the coefficient of $(1/2)\phi_i(-\partial^2)\phi_i$ is

$$-\partial_g B(0; 0) = 0 \tag{C.16}$$

These are consequences of the conventions (56a, 56b).

Appendix C.3. $\mathcal{N}$

Using the interaction part of the action, we can rewrite

$$\mathcal{N} = \int_p p^2 \phi_i(p)\phi_i(-p) + \int_p \left\{-1 + 2\left(1 - K\left(p/\Lambda\right)\right)\right\} \phi_i(p) \frac{\delta S_{I,\Lambda}}{\delta \phi_i(p)}$$
$$- \int_p \frac{K\left(p/\Lambda\right)\left(1 - K\left(p/\Lambda\right)\right)}{p^2} \left\{\frac{\delta S_{I,\Lambda}}{\delta \phi_i(p)} \frac{\delta S_{I,\Lambda}}{\delta \phi_i(-p)} + \frac{\delta^2 S_{I,\Lambda}}{\delta \phi_i(p)\delta \phi_i(-p)}\right\} \quad \text{(C.17)}$$

Hence, the derivative expansion gives

$$\mathcal{N}(\Lambda) = \int d^2 x \left[\Lambda^2 a_\mathcal{N}\left(\ln \Lambda/\mu; \phi^2/2\right) + A_\mathcal{N}\left(\ln \Lambda/\mu; \phi^2/2\right)\left(-\partial^2\right)\frac{\phi^2}{2}\right.$$
$$\left. + B_\mathcal{N}\left(\ln \Lambda/\mu; \phi^2/2\right)\phi_i\left(-\partial^2\right)\phi_i\right] + \cdots \quad \text{(C.18)}$$

where

$$A_\mathcal{N}\left(\ln \Lambda/\mu; x\right) = -2A\left(\ln \Lambda/\mu; x\right) - 2x\frac{\partial}{\partial x}A\left(\ln \Lambda/\mu; x\right) + \cdots \quad \text{(C.19)}$$
$$B_\mathcal{N}\left(\ln \Lambda/\mu; x\right) = 1 - 2B\left(\ln \Lambda/\mu; x\right) + \cdots \quad \text{(C.20)}$$

up to loop corrections. At $\Lambda = \mu$, the coefficient of $(\phi^2/2)(-\partial^2)(\phi^2/2)$ is

$$\partial_x A_\mathcal{N}(0; x)\Big|_{x=0} = 2g + \cdots \quad \text{(C.21)}$$

and the coefficient of $(1/2)\phi_i(-\partial^2)\phi_i$ is

$$B_\mathcal{N}(0; 0) = 1 + \cdots \quad \text{(C.22)}$$

Appendix C.4. β and γ

Comparing the derivative expansion of Ψ_Λ with those of $\mathcal{O}_g$ and $\mathcal{N}$, we obtain $\beta(g)$ and $\gamma(g)$ as follows:

$$\partial_x \dot{A}(0; x)\Big|_{x=0} = \frac{1}{2}\beta(g) + \gamma(g)\partial_x A_\mathcal{N}(0; x)\Big|_{x=0} \quad \text{(C.23)}$$
$$\dot{B}(0; 0) = \gamma(g)B_\mathcal{N}(0; 0) \quad \text{(C.24)}$$

where we have used (C.15, C.16).

Using (58b, 57b), we obtain

$$\partial_x \dot{A}^{(1)}(0; x)\Big|_{x=0} = N\frac{g^2}{4\pi}, \quad \dot{B}^{(1)}(0; 0) = \frac{g}{4\pi} \quad \text{(C.25)}$$

Using (C.21, C.22), we also obtain

$$\partial_x A_\mathcal{N}^{(0)}(0; x)\Big|_{x=0} = 2g, \quad B_\mathcal{N}^{(0)}(0; 0) = 1 \quad \text{(C.26)}$$

Hence, at 1-loop (C.23, C.24) give

$$N\frac{g^2}{4\pi} = \frac{1}{2}\beta^{(1)} + \gamma^{(1)} \cdot 2g \quad \text{(C.27)}$$
$$\frac{g}{4\pi} = \gamma^{(1)} \cdot 1 \quad \text{(C.28)}$$

Thus, we obtain (64).

References

[1] A. M. Polyakov, Phys. Lett. B **59**, 79 (1975).

[2] C. G. Bollini and J. J. Giambiagi, Nuovo Cim. B **12**, 20 (1972).

[3] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **44**, 189 (1972).

[4] K. G. Wilson and J. B. Kogut, Phys. Rept. **12**, 75 (1974).

[5] J. Polchinski, Nucl. Phys. B **231**, 269 (1984).

[6] Y. Igarashi, K. Itoh, and H. Sonoda, "Realization of symmetry in the ERG approach to quantum field theory," arXiv:0909.0327 [hep-th].

[7] C. Becchi, "On the construction of renormalized gauge theories using renormalization group techniques," arXiv:hep-th/9607188.

[8] M. Pernici and M. Raciti, Nucl. Phys. B **531**, 560 (1998) [arXiv:hep-th/9803212].

[9] H. Sonoda, Phys. Rev. D **67**, 065011 (2003) [arXiv:hep-th/0212302].

[10] H. Sonoda, J. Phys. A **40**, 5733 (2007) [arXiv:hep-th/0612294].

[11] G. 't Hooft, Nucl. Phys. B **61**, 455 (1973).